A

NATIONAL BOARD FOR HIGHER MATHEMATICS

AND

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2013

Mumbai Region

October 20, 2013

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2 hours.

QUESTIONS

1. What is the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 1$?

2. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1} + \sqrt{k}}$. What is the value of $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$?

3. It is given that the equation $x^2 + ax + 20 = 0$ has integer roots. What is the sum of all possible values of a?

4. Three points X, Y, Z are on a striaght line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ?

5. There are $n-1$ red balls, n green balls and $n+1$ blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n?

6. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$?

7. Let Akbar and Birbal together have n marbles, where $n > 0$.

   Akbar says to Birbal, " If I give you some marbles then you will have twice as many marbles as I will have." Birbal says to Akbar, " If I give you some marbles then you will have thrice as many marbles as I will have."

   What is the minimum possible value of n for which the above statements are true?

8. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the midpoints of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ?

A

9. In a triangle ABC, let H, I and O be the orthocentre, incentre and circumcentre, respectively. If the points B, H, I, C lie on a circle, what is the magnitude of $\angle BOC$ in degrees?

10. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers?

11. Three real numbers x, y, z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$ and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$?

12. Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3}PS$ and the area of $PQRS$ is $28\sqrt{3}$, what is the length of PC?

13. To each element of the set $S = \{1, 2, \ldots, 1000\}$ a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides $a + b$ then they are both assigned the same colour. What is the maximum possible number of distinct colours used?

14. Let m be the smallest odd positive integer for which $1 + 2 + \cdots + m$ is a square of an integer and let n be the smallest even positive integer for which $1 + 2 + \cdots + n$ is a square of an integer. What is the value of $m + n$?

15. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$?

16. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$, where b is a real number. What is the sum of all possible values of b for which the equations $f(x) = 0$ and $g(x) = 0$ have a common root?

17. Let S be a circle with centre O. A chord AB, not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be a circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point X lies on the diameter passing through the centre of S_2 and $\angle YXO = 30°$. If the radius of S_2 is 100 then what is the radius of S_1?

18. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers?

19. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is $13 : 18$ and $BC = 12$ then what is the length of AC?

20. What is the sum (in base 10) of all the natural numbers less than 64 which have exactly three ones in their base 2 representation?